Section 1.1: Points and Vectors in $\mathbb{R}^n$

- A **point** in $\mathbb{R}^n$ is an ordered sequence of real numbers, denoted

$$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

- A **vector** in $\mathbb{R}^n$ is an ordered sequence of real numbers, denoted

$$\langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n.$$

- Generally, a point signifies a location, while a vector signifies direction and magnitude.
- However, this distinction will be blurred somewhat, since we will use the "position vector of a point" to represent the point itself. The position vector of the point

$$(x_1, x_2, \ldots, x_n)$$

is

$$\langle x_1, x_2, \ldots, x_n \rangle.$$

Section 1.2: Vector Operations

- To add two vectors in $\mathbb{R}^n$, we add coordinates:

$$\langle x_1, x_2, \ldots, x_n \rangle + \langle y_1, y_2, \ldots, y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n \rangle.$$

- To multiply a vector by a scalar α, we multiply each coordinate by α:

$$\alpha \cdot \langle x_1, x_2, \ldots, x_n \rangle = \langle \alpha x_1, \alpha x_2, \ldots, \alpha x_n \rangle.$$

- Except in special cases (namely, the cross product in $\mathbb{R}^3$), there is no natural notion of "vector multiplication."

Properties of Vector Operations

- **Theorem 1.8:**
  1. For all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$, $\vec{\mathbf{x}} + \vec{\mathbf{y}} = \vec{\mathbf{y}} + \vec{\mathbf{x}}$.
  2. For all $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} \in \mathbb{R}^n$, $(\vec{\mathbf{x}} + \vec{\mathbf{y}}) + \vec{\mathbf{z}} = \vec{\mathbf{x}} + (\vec{\mathbf{y}} + \vec{\mathbf{z}})$.
  3. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$, $\vec{\mathbf{x}} + \vec{0} = \vec{0} + \vec{\mathbf{x}} = \vec{\mathbf{x}}$.
  4. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$, $\vec{\mathbf{x}} + (-\vec{\mathbf{x}}) = (-\vec{\mathbf{x}}) + \vec{\mathbf{x}} = \vec{0}$.
  5. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$, $\alpha(\beta\vec{\mathbf{x}}) = (\alpha\beta)\vec{\mathbf{x}}$.
  6. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$, $(\alpha + \beta)\vec{\mathbf{x}} = \alpha\vec{\mathbf{x}} + \beta\vec{\mathbf{x}}$.
  7. For all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$, $\alpha(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = \alpha\vec{\mathbf{x}} + \alpha\vec{\mathbf{y}}$.
  8. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$, $1\vec{\mathbf{x}} = \vec{\mathbf{x}}$.
  9. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$, $0\vec{\mathbf{x}} = \vec{0}$.

Length of a Vector

- Given a vector $\vec{\mathbf{x}} = \langle x_1, \ldots, x_n \rangle$ in $\mathbb{R}^n$, the **magnitude** or **length** of $\vec{\mathbf{x}}$ is defined as

$$\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

- The distance between points A and B is $\left\| \overrightarrow{AB} \right\|$.
- **Theorem 1.10:**
  1. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$, $\|\vec{\mathbf{x}}\| \geq 0$.
  2. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$, $\|\vec{\mathbf{x}}\| = 0$ if and only if $\vec{\mathbf{x}} = \vec{0}$.
  3. For all $\vec{\mathbf{x}} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $\|\alpha\vec{\mathbf{x}}\| = |\alpha| \, \|\vec{\mathbf{x}}\|$.

Unit Vectors

- A vector whose length is 1 is called a unit vector.
- A nonzero vector $\vec{\mathbf{x}} \in \mathbb{R}^n$ can be scaled to a unit vector $\vec{\mathbf{u}}$ in the same direction as $\vec{\mathbf{x}}$ using the formula

$$\vec{\mathbf{u}} = \frac{1}{\|\vec{\mathbf{x}}\|}\vec{\mathbf{x}}.$$